Quasi-dilaton non-linear massive gravity: Investigations of background cosmological dynamics

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We investigate cosmological behavior in the quasi-Dilaton non-linear massive gravity. We perform a detailed dynamical analysis and examine the stable late-time solutions relevant to late time cosmic acceleration. We demonstrate that a dark-energy dominated, cosmological-constant-like solution is a late attractor of the dynamics. We also analyze the evolution of the universe at intermediate times, showing that the observed epoch sequence can be easily obtained in the model under consideration. Furthermore, we study the non-singular bounce and the cosmological turnaround which can be realized for a region of the parameter space. Last but not least, we use observational data from Type Ia Supernovae (SNIa), Baryon Acoustic Oscillations (BAO) and Cosmic Microwave Background (CMB), in order to constrain the parameters of the theory.

I. INTRODUCTION

It is widely believed that an elementary particle of mass m and spin s is described by a field which transforms according to a particular representation of Poincaré group. In field theory, formulated in Minkowski space-time, mass can either be introduced by hand or via spontaneous symmetry breaking. The spin-2 field $h_{\mu\nu}$ is supposed to be relevant to gravity as it shares an important property of universality with Einstein general relativity(GR) *a la Weinberg theorem* and GR can be thought as an interacting theory of $h_{\mu\nu}$. It is therefore natural to first formulate the field theory of $h_{\mu\nu}$ in Minkowski space-time and then extend the concept to non-linear background.

The first formulation of a linear massive theory of gravity was given by Fierz and Pauli [1] with a motivation to write down the consistent relativistic equations for higher spin fields including spin-2 field. The linear theory of Fierz-Pauli, however, suffers from van Dam, Veltman, Zakharov (vDVZ) discontinuity [2, 3]: in the zero-mass limit, the theory is at finite difference from GR. The underlying reason of the (vDVZ) discontinuity is related to the fact the longitudinal degree of freedom of massive graviton does not decouple in the said limit; it is rather coupled to the source with the strength of the universal coupling at par with the massless mode.

It was pointed out by Vainshtein that linear approximation in the neighborhood of a massive source breaks down below certain distance dubbed Vainshtein radius and by incorporating non-linearities one could remove the discontinuity [4] present in the linear theory. However, the sixth degree of freedom, suppressed in Pauli-Fierz theory, which is essentially a ghost known as Boulware-Deser (BD) ghost [5], becomes alive in the non-linear

background.

Efforts were then made to find out the non-linear analog of Fierz-Pauli mass term requiring the ghost to be systematically removed. The authors of Ref.[6, 7] discovered a specific nonlinear extension of massive gravity (*dRGT*) that is BD-ghost free in the decoupling limit(see [8] for a review). Subsequently, it was demonstrated [9] that the Hamiltonian constraint is maintained at the non-linear order along with the associated secondary constraint, which implies the absence of the BD ghost. Apart from the field theoretic interests of such a construction, there is a cosmological motivation to study massive gravity. This formulation provides a new class of (Infra-Red) gravity modification which can give rise to late-time cosmic acceleration and it is therefore not surprising that massive gravity has recently generated enormous interest in cosmology [9–71].

In spite of the the successes of the framework, it has challenging problems to be resolved, namely, the model does not admit spatially flat FLRW background whereas isotropic and homogeneous cosmological solutions ($K = \pm 1$) are perturbatively unstable [72]. The latter speaks of some inherent difficulty of *dRGT*. This led the authors of [73, 74] to abandon isotropy whereas in Ref.[75], effort was made to address the problem by making the graviton mass a field dependent quantity through a more radical approach, namely by extending the theory to a varying graviton mass driven by a scalar field [76, 77]. A similar approach was followed in [78], where the framework of massive gravity was extended by introducing a quasi-dilaton field. It was pointed that the model could rise to a healthy cosmology in the FLRW

background[1].

In this work we carry out detailed dynamical investigations of cosmological behaviors in the aforementioned quasi-Dilaton non-linear massive gravity. In particular, we perform dynamical analysis relevant to late time cosmic acceleration, and moreover we examine the realization of non-singular bouncing solutions. Finally, we use observational data in order to constrain the model parameters.

The plan of the paper is as follows: In section II we review the quasi-dilaton non-linear massive gravity and its cosmological implications. In section III, we perform a detailed dynamical analysis and obtain the late-time stable solutions, while in section IV we investigate the universe evolution focusing on bouncing and turnaround solutions. In section V we use observations in order to constrain the model parameters and finally in section VI, we summarize the results of our analysis.

II. QUASI-DILATON NON-LINEAR MASSIVE GRAVITY AND COSMOLOGY

In this section we briefly review the quasi-dilaton non-linear massive gravitational formulation following [78]. In the first subsection we present the gravitational theory itself, while in the next subsection we examine its cosmological implications in presence of matter and radiation.

A. Massive gravity with quasi-dilaton

Let us consider the following action of non-linear massive gravity with dilaton

$$\mathcal{S}_{\text{gr}} = \mathcal{S}_{\text{EH}} + \mathcal{S}_{\text{mass}} + \mathcal{S}_{\sigma}, \qquad (1)$$

with

$$\mathcal{S}_{\text{EH}} = \frac{M_{Pl}^2}{2} \int \mathrm{d}^4x \sqrt{-g} \, R \qquad (2)$$

the usual Einstein-Hilbert action,

$$\mathcal{S}_{\text{mass}} = \frac{m^2 M_{Pl}^2}{8} \int \mathrm{d}^4x \sqrt{-g} \left[U_2 + \alpha_3 U_3 + \alpha_4 U_4 \right] \quad (3)$$

the action corresponding to non-linear massive term, along with the action of a massless dilaton σ

$$\mathcal{S}_{\sigma} = -\frac{\omega}{2} \int \mathrm{d}^4x \, (\partial\sigma)^2. \qquad (4)$$

In action (3), α_3, α_4 are two arbitrary parameters and U_i are specific polynomials of the matrix

$$\mathcal{K}_{\nu}^{\mu} = \delta_{\nu}^{\mu} - e^{\sigma/M_{Pl}} \sqrt{g^{\mu\alpha} \partial_{\alpha}\phi^a \partial_{\nu}\phi^b \eta_{ab}}, \qquad (5)$$

given by

$$U_2 = 4([\mathcal{K}]^2 - [\mathcal{K}^2]) \qquad (6)$$

$$U_3 = [\mathcal{K}]^3 - 3[\mathcal{K}][\mathcal{K}^2] + 2[\mathcal{K}^3] \qquad (7)$$

$$U_4 = [\mathcal{K}]^4 - 6[\mathcal{K}]^2[\mathcal{K}^2] + 3[\mathcal{K}^2]^2 + 8[\mathcal{K}][\mathcal{K}^3] - 6[\mathcal{K}^4]. \quad (8)$$

In (5), η_{ab} (Minkowski metric) is a fiducial metric, and $\phi^a(x)$ are the Stückelberg scalars introduced to restore general covariance [81].

B. Cosmology

In order to apply the above gravitational theory in cosmological frameworks, we also have to incorporate the matter and radiations sectors. Thus, the total action we consider is as follows

$$\mathcal{S} = \mathcal{S}_{\text{EH}} + \mathcal{S}_{\text{mass}} + \mathcal{S}_{\sigma} + \mathcal{S}_m + \mathcal{S}_r, \qquad (9)$$

in which $\mathcal{S}_m$ and $\mathcal{S}_r$ denote the standard matter and radiation actions, corresponding to ideal fluids with energy densities ρ_m, ρ_r and pressures p_m, p_r respectively. In summary, we consider a scenario of usual nonlinear massive gravity [6, 7] coupled with a dilaton field σ, where the coupling is introduced through (5) [78].

Next, we consider a flat Friedmann-Lemaître-Robertson-Walker (FLRW) metric of the form

$$\mathrm{d}s^2 = g_{\mu\nu}dx^\mu dx^\nu = -N(t)^2 dt^2 + a^2(t)\delta_{ij}dx^i dx^j, \quad (10)$$

while for the Stückelberg scalars we consider the ansatz

$$\phi^0 = f(t), \quad \phi^i = x^i. \qquad (11)$$

In this case the actions (2)-(4) become:

$$\mathcal{S}_{\text{EH}} = -3M_{Pl}^2 \int \mathrm{d}t \left[\frac{a\dot{a}^2}{N} \right] \qquad (12)$$

$$\mathcal{S}_{\text{mass}} = 3m^2 M_{Pl}^2 \int \mathrm{d}t\, a^3 \left[NG_1(\xi) - \dot{f}aG_2(\xi) \right] \qquad (13)$$

$$\mathcal{S}_{\sigma} = \frac{\omega}{2} \int \mathrm{d}t\, a^3 \left[\frac{\dot{\sigma}^2}{N} \right], \qquad (14)$$

where we have defined

$$G_1(\xi) = (1-\xi)\left[2 - \xi + \frac{\alpha_3}{4}(1-\xi)(4-\xi) + \alpha_4(1-\xi)^2 \right] \qquad (15)$$

$$G_2(\xi) = \xi(1-\xi)\left[1 + \frac{3}{4}\alpha_3(1-\xi) + \alpha_4(1-\xi)^2 \right] \qquad (16)$$

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[1] During the finalization of the present work, a perturbation analysis of the stability of the model was performed in [79, 80], where it was shown that the de Sitter solution of the model may have a ghost instability for short wavelength modes, shorter than cosmological scales. The calculations were performed at the linear order of perturbations, or equivalently at second order in the action. It will be interesting to see whether the ghost exists in a non-linear approach of the stability analysis.

being functions of the introduced dilaton-variable

$$\xi = \frac{e^{\sigma/M_{Pl}}}{a}. \qquad (17)$$

Variation with respect to f gives the constraint equation (as an essential feature of any model of non-linear massive gravity)

$$G_2(\xi) = \frac{C}{a^4}, \qquad (18)$$

where C is a constant of integration.

Variation with respect to the lapse function N and setting $N = 1$ at the end, leads to the first Friedmann equation

$$3M_{Pl}^2 H^2 = \frac{\rho_m + \rho_r - 3m^2 M_{Pl}^2 G_1}{1 - \frac{\omega}{6}\left(1 - 4\frac{G_2}{\xi G_2'}\right)^2}, \qquad (19)$$

where primes in G's denote derivatives with respect to their argument ξ. Similarly, variation with respect to σ provides the dilaton evolution equation

$$\frac{\omega}{a^3}\frac{d}{Ndt}(a^3\dot\sigma) + 3M_{Pl}m^2\left[F_1(\xi) + a\dot f F_2(\xi)\right] = 0, \quad (20)$$

where

$$
\begin{aligned}
F_1(\xi) &= 3(1 + \frac{3}{4}\alpha_3 + \alpha_4)\xi - 2(1 + \frac{3}{2}\alpha_3 + 3\alpha_4)\xi^2 \\
&\quad + \frac{3}{4}(\alpha_3 + 4\alpha_4)\xi^3 \qquad (21)
\end{aligned}
$$

$$
\begin{aligned}
F_2(\xi) &= (1 + \frac{3}{4}\alpha_3 + \alpha_4)\xi - 2(1 + \frac{3}{2}\alpha_3 + 3\alpha_4)\xi^2 \\
&\quad + \frac{9}{4}(\alpha_3 + 4\alpha_4)\xi^3 - 4\alpha_4\xi^4. \qquad (22)
\end{aligned}
$$

Finally, the evolution equations close by considering the continuity equations for matter and radiation, namely $\dot\rho_m + 3H(\rho_m + p_m) = 0$ and $\dot\rho_r + 3H(\rho_r + p_r) = 0$ respectively.

Before proceeding to the detailed analysis of the model, let us make a few comments. Firstly, as it was first noticed for a similar model in [67], the Friedmann equation (19) can exhibit singularities when the denominator goes to zero. However, as we will show in the next section, these singularities separate the phase space in disconnected branches, that is the universe cannot evolve from one to the other.

As a second remark we mention that in the case where $C \neq 0$, in the constraint equation (18) it appears a function of $1/a^4$ which plays, in the early Universe, a role similar to "Dark Matter" or "Dark Radiation". In particular, when $\alpha_4 \neq 0$, in the early Universe ($a \to 0$) relation (17) gives $\xi \to \infty$. This implies $G_2 \simeq -\alpha_4\xi^4 = C/a^4$ leading the Friedmann equation (19) to be

$$3M_{Pl}^2 H^2 \simeq \rho_m + \rho_r + 3m^2 M_{Pl}^2\left(\alpha_4 + \frac{\alpha_3}{4}\right)\left(-\frac{C}{\alpha_4}\right)^{3/4}a^{-3}, \qquad (23)$$

that is an effective dark-matter term appears. Similarly, when $\alpha_4 = 0$ we acquire

$$3\left(1 - \frac{\omega}{54}\right)M_{Pl}^2 H^2 \simeq \rho_m + \rho_r + m^2 M_{Pl}^2 Ca^{-4}, \qquad (24)$$

that is we get an effective dark radiation term.

Finally, we mention that in the particular case $C = 0$, (18) leads to a constant ξ, and the Friedmann equation (19) becomes

$$3M_{Pl}^2 H^2 = \rho_m + \rho_r - \alpha_\Lambda + \beta H^2, \qquad (25)$$

where α_Λ, β are constants. It is interesting to notice that this relation appears as a particular case of Holographic models where the IR cut-off is fixed at the Hubble scale. In fact in the aforementioned theories the number of degrees of freedom in a finite volume should be finite and related to the area of its boundary. This gives an upper bound on the entropy contained in the visible Universe. Following an idea suggested by [82, 83] where a long distance cut-off is related to a short distance cut-off, in [84] it was suggested to take the largest distance as the Hubble scale or the event horizon. Later these models were dubbed holographic dark energy [85].

All these particular cases reveal the richness and the capabilities of the scenario. In what follows we shall cast the evolution equations in the autonomous form and investigate their implications for late time cosmology.

III. DYNAMICAL BEHAVIOR

In this section we perform a dynamical analysis of the scenario at hand, which allows us to bypass the complexities of the equations and obtain information for the late-time, asymptotic, cosmological behavior [86–88]. We mention here that in [78] the authors investigated the cosmological implications in a specific setting allowing them to obtain simple analytical solutions. However, in this work we are interested in the full dynamical investigations of evolution equations.

A. Phase-space analysis

In order to perform a phase-space analysis, we first transform the involved cosmological equations into their autonomous form introducing suitable auxiliary variables. We extract the critical points of the dynamical system under consideration, perturb the system around them and analyze their stability by examining the eigenvalues of the corresponding perturbation matrix.

Let us firs introduce the density parameters

$$\Omega_m = \frac{\rho_m}{3M_{Pl}^2 H^2} \qquad (26)$$

$$\Omega_r = \frac{\rho_r}{3M_{Pl}^2 H^2} \qquad (27)$$

$$\Omega_\Lambda = -m^2 \frac{G_1}{H^2} \qquad (28)$$

$$\Omega_\sigma = \frac{\omega}{6}\left(1 - 4\frac{G_2}{\xi G_2'}\right)^2, \qquad (29)$$

with which we can re-write the Friedmann equation (19) as

$$\Omega_m + \Omega_r + \Omega_\Lambda + \Omega_\sigma = 1. \qquad (30)$$

Thus, we can now transform the above cosmological system into its autonomous form, using only the dimensionless variables Ω_r, Ω_Λ and ξ, while (30) will be used to eliminate Ω_m. Doing so we obtain

$$\frac{d\Omega_r}{d\ln a} = -2\Omega_r\left(2 + \frac{\dot{H}}{H^2}\right) \qquad (31)$$

$$\frac{d\Omega_\Lambda}{d\ln a} = -2\Omega_\Lambda\left(2\frac{G_2 G_1'}{G_1 G_2'} + \frac{\dot{H}}{H^2}\right) \qquad (32)$$

$$\frac{d\xi}{d\ln a} = -4\frac{G_2}{G_2'}, \qquad (33)$$

where the combination $\frac{\dot{H}}{H^2}$ can be acquired differentiating the first Friedmann equation (19) as

$$\frac{\dot{H}}{H^2} = \frac{-9\Omega_m - 12\Omega_r - 12\frac{G_2}{G_2'}\left[\frac{G_1'}{G_1}\Omega_\Lambda + \frac{\omega}{6}\frac{d}{d\xi}(1 - 4\frac{G_2}{\xi G_2'})^2\right]}{6 - \omega\left[1 - 4\frac{G_2}{\xi G_2'}\right]^2}. \qquad (34)$$

A comment about the role of dilaton in the scenario under consideration is in order. The dilaton role is twofold, first, it has an effect through its impact on the graviton mass, since in the above equations m appears multiplied by a function of ξ, that is of the exponential of the dilaton field. This is what we call Ω_Λ. Secondly, the dilaton has an effect through its kinetic term in (14), which is of course proportional to the parameter ω. This is what we call Ω_σ. Therefore, strictly speaking, in the present scenario the effective dark energy sector includes both Ω_Λ and Ω_σ, since these two contributions cannot be distinguished observationally (we write them separately only for clarity). Thus:

$$\Omega_{DE} \equiv \Omega_\Lambda + \Omega_\sigma = 1 - \Omega_m - \Omega_r. \qquad (35)$$

Finally, note that in terms of the auxiliary variables Ω_r, Ω_Λ and ξ, we can express the physically interesting observables such as the total (effective) equation-of-state parameter w_{eff}, the deceleration parameter q and the dark energy equation-of-state parameter w_{DE} respectively as

$$w_{eff} = -1 - \frac{2}{3}\frac{\dot{H}}{H^2} \qquad (36)$$

$$q = -1 - \frac{\dot{H}}{H^2} = \frac{1}{2} + \frac{3}{2}w_{eff}. \qquad (37)$$

$$w_{DE} = \frac{w_{eff} - w_m\Omega_m - w_r\Omega_r}{\Omega_\Lambda + \Omega_\sigma}, \qquad (38)$$

where the combination $\dot{H}/H^2$ is given by (34). Lastly, in the following for simplicity we focus on the usual cases of standard dust matter $w_m = 0$ and standard radiation $w_r = 1/3$.

The critical points of the above autonomous system are extracted setting the left hand sides of equations (31)-(33) to zero. In particular, equation (33) implies that at the critical points $G_2(\xi) = 0$, which according to (16) gives $\xi = 0, \xi = 1$ and

$$\xi_\pm = 1 + \frac{3\alpha_3 \pm \sqrt{9\alpha_3^2 - 64\alpha_4}}{8\alpha_4}. \qquad (39)$$

In order for the critical points to be physical they have to possess $0 \le \Omega_r \le 1$, $0 \le \Omega_\Lambda + \Omega_\sigma \le 1$, $0 \le \xi$ and of course $\xi \in \mathcal{R}$. Now, in order for $\xi_\pm \in \mathcal{R}$ we require $9\alpha_3^2 - 64\alpha_4 \ge 0$ while $0 \le \xi$ leads to the necessary conditions for α's using (39).

The real and physically meaningful critical points $(\Omega_r, \Omega_\Lambda, \xi)$ of the autonomous system (31)-(33) are presented in Table I, along with their existence conditions. We mention here that there is an additional existence condition, namely the obtained H^2 from (19) must be always positive, a condition which constrains the allowed α_3 and α_4 values. We will come back to this later on.

For each critical point we calculate the 3×3 perturbation matrix of the corresponding linearized perturbation equations, and examining the sign of the real part of its eigenvalues we determine the type and stability of this point (positive real parts correspond to unstable point, negative real parts correspond to stable point, and real parts with different signs correspond to saddle point). The eigenvalues and the stability results are summarized in Table I.

Finally, for completeness, in the same Table we present the corresponding values of the rest density parameters Ω_m and Ω_σ, as well as the calculated values for the observables such as the deceleration parameter q, the total equation-of-state parameter w_{eff}, and the dark energy equation-of-state parameter w_{DE}. These results are in agreement with the general analysis of nonlinear massive gravity of [89].

Before proceeding to the discussion of the physical implications of these results, we make a remark concerning the structure of the phase space. As we mentioned in the Introduction, the Friedmann equation (19) exhibits singularities when the denominator goes to zero. In particular, in the scenario at hand we obtain various singularities localized at the roots of $1 - \frac{\omega}{6}\left(1 - 4\frac{G_2}{\xi G_2'}\right)^2$, which, for every parameter ω, is a function of the rescaled field-variable ξ only. Since ω must be relatively small in order to satisfy observations (this will be confirmed later on), assuming

Cr.P.	Ω_r	Ω_Λ	ξ	Ω_m	Ω_σ	q	w_{eff}	w_{DE}	Existence conditions	Stability	Eigenvalues
A	0	0	0	$1-3\omega/2$	$3\omega/2$	1/2	0	0	$0 \le \omega \le 2/3$	Saddle point	-4, 3, -1
B	0	$1-3\omega/2$	0	0	$3\omega/2$	-1	-1	-1	$0 \le \omega \le 2/3$	Attractor	-4, -4, -3
C	$1-3\omega/2$	0	0	0	$3\omega/2$	1	1/3	1/3	$0 \le \omega \le 2/3$	Saddle point	-4, 4, 1
D	0	0	1	$1-\omega/6$	$\omega/6$	1/2	0	0	$0 \le \omega \le 6$	Attractor	-4, -1, -1
$E_\pm$	0	0	$\xi_\pm$	$1-\omega/6$	$\omega/6$	1/2	0	0	$0 \le \omega \le 6,\ 0 \le \xi_\pm,\ \xi_\pm \in \mathcal{R}$	Saddle point	-4,-1, 3
$F_\pm$	$1-\omega/6$	0	$\xi_\pm$	0	$\omega/6$	1	1/3	1/3	$0 \le \omega \le 6,\ 0 \le \xi_\pm,\ \xi_\pm \in \mathcal{R}$	Saddle point	-4,4,1
$G_\pm$	0	$1-\omega/6$	$\xi_\pm$	0	$\omega/6$	-1	-1	-1	$0 \le \omega \le 6,\ 0 \le \xi_\pm,\ \xi_\pm \in \mathcal{R}$	Attractor	-4,-4,-3
I	Ω_r	$1-\omega/6-\Omega_r$	1	0	$\omega/6$	1	1/3	1/3	$0 \le \omega \le 6$	Saddle	-4,1,0

TABLE I: The real and physically meaningful critical points of the autonomous system (31)-(33), their existence conditions, the eigenvalues of the perturbation matrix, and the deduced stability conditions. We also present the corresponding values of the various density parameters, and the values of the observables: deceleration parameter q, total equation-of-state parameter w_{eff} and dark energy equation-of-state parameter w_{DE}.

$\omega \ll 1$ the zeros of the previous function nearly coincide with the roots of $G_2'(\xi)$. But as we mentioned above, the critical points correspond to $G_2(\xi) = 0$. Thus, using the Rolle's theorem, we conclude that each cosmological late-time solution ($G_2(\xi) = 0$) and the corresponding region, is disconnected from the others by a singularity ($G_2' = 0$). In other words, during its evolution, the universe cannot transit from one of these regions to another.

B. Physical implications

In the previous subsection we performed a complete phase-space analysis of the scenario at hand, we extracted the late-time stable solutions (attractors) and we calculated the corresponding observables. Here we discuss their cosmological implications.

As we observe from Table I, there exist four stable critical points, that can attract the universe at late times, namely B, D and the two points $G_\pm$. Points B and $G_\pm$ correspond to a dark-energy dominated, accelerating universe, where dark energy behaves like cosmological constant (de Sitter solutions). These features make these points very good candidates for the description of late-time universe. On the other hand, point D corresponds to a non-accelerating universe, with dark energy and matter density parameters of the same order (that is it can alleviate the coincidence problem), but the dark energy has a stiff equation of state. Therefore, this point is disfavored by observations.

Apart from the above stable late-time solutions, the scenario of quasi-Dilaton non-linear massive gravity possesses the non-accelerating, stiff dark-energy points $A,C,E_\pm,F_\pm$, and the critical curve I. These points are saddle (I is non-hyperbolic curve with saddle behavior) and thus they cannot be the late time states of the universe, and moreover their cosmological features are not favored by observations.

Let us make a comment here on the appearance of more than one stable critical points, which for some parameter choices can exist simultaneously. Although the bi-stability and multi-stability is usual in complicated dynamical systems [90, 91], with each critical point attracting orbits initially starting inside its basin of attraction, in the present scenario the case is simpler, since these points usually lie in the disconnected regions we mentioned in the previous subsection, separated by the ξ values that make the denominator of (19) zero, and thus to Big-Rip-type divergences [92–95]. Finally, note that for $\omega > 6$ there are not stable late-time solutions, that is the corresponding cosmology is not interesting.
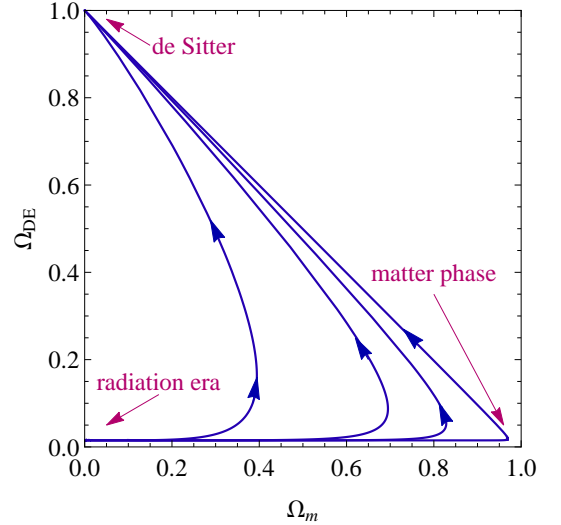

FIG. 1: *Trajectories in the Ω_{DE}-Ω_m plane of the cosmological scenario of quasi-Dilaton non-linear massive gravity, for the parameter choice $\alpha_3 = 1, \alpha_4 = 1$ and $\omega = 0.01$. In this specific example the universe at late times is led to the de Sitter solution.*

Finally, as we previously mentioned, the α_3-α_4 parameter subspace is constrained by the condition the obtained H^2 from (19) to be always positive. The corresponding constraints are complicated and we do not present them here, however they do get a very simple form when the universe evolves towards its interesting stable late-time states B and $G_\pm$ (where Ω_r and Ω_m become smaller and smaller). In particular, for the point B

we obtain $2 + \alpha_3 + \alpha_4 < 0$, for the point G_+ we acquire $\alpha_3 > 0$, $0 < \alpha_4 < \frac{\alpha_3^2}{8}$ and for the point G_- we obtain $\alpha_3 < 0$ and $0 < \alpha_4 < \frac{\alpha_3^2}{8}$.

In order to present the cosmological behavior more transparently, we numerically evolve the autonomous system (31)-(33) for the the parameter choice $\alpha_3 = 1$, $\alpha_4 = 1$ and $\omega = 0.01$, and in Fig. 1 we depict the corresponding phase-space behavior in the Ω_{DE}-Ω_m plane.

IV. COSMOLOGICAL EVOLUTION

In the previous section we performed a dynamical analysis of quasi-Dilaton non-linear massive gravity, that is we focused on its asymptotic behavior at late times, that is on solutions that are going to attract the universe independently of the initial conditions and of the specific cosmological evolution towards them. In this section we investigate the behavior of the universe at intermediate times, which obviously does depend on the initial conditions, but it can be very interesting. In particular, we are interested in obtaining a evolution of the universe in agreement with the observed epoch sequence. Furthermore, we examine the possibility of obtaining bouncing behavior. In the following subsections we discuss these two cosmological behaviors separately.

A. Epoch sequence and late-time acceleration

Let us first examine the universe evolution, focusing on the various density parameters and the dark-energy equation of state. Observing the Friedmann equation (19), as well as the evolution equations for the density parameters and for ξ (31)-(33), we deduce that the observed post-inflationary thermal history of the universe can be easily obtained by suitably choosing the model parameters.

In order to present this behavior more transparently, we numerically evolve the system for $\alpha_3 = 1$, $\alpha_4 = 0.115$ and $\omega = 0.01$, starting with initial conditions corresponding to $\Omega_r \approx 1$, and imposing the current values to be $\Omega_m \approx 0.28$ and $\Omega_{DE} \approx 0.72$. In Fig. 2 we depict the resulting evolution for the density parameters using, instead of the scale factor, the redshift z as the independent variable ($1 + z = a_0/a$ with $a_0 = 1$ the present scale-factor value). As we observe, the thermal history of the universe can be reproduced, namely we obtain the successive sequence of radiation, matter and dark energy epochs.

Similarly, in Fig. 3 we present the corresponding evolution of the total equation-of-state parameter w_{eff} as well as of the dark-energy one w_{DE}. As we observe, dark energy starts from a dust-like behavior, resulting in a cosmological-constant-like one at the current universe, driving the observed acceleration. Note that, as expected, in the future, where dark energy completely
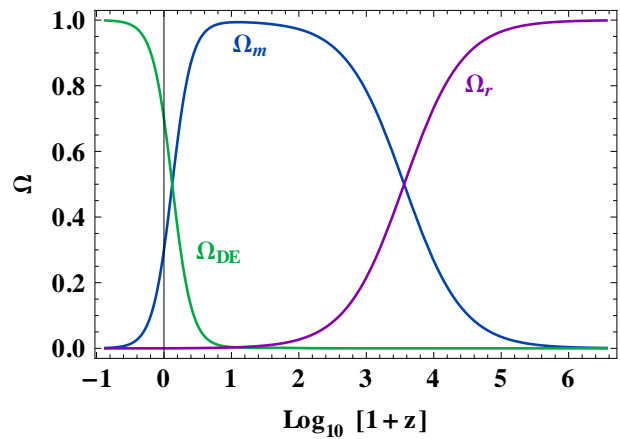

FIG. 2: *Cosmological evolution of the radiation (Ω_r), matter (Ω_m) and dark energy ($\Omega_{DE} \equiv \Omega_\Lambda + \Omega_\sigma$) density parameters, as a function of the redshift z, for the parameter choice $\alpha_3 = 1$, $\alpha_4 = 0.115$ and $\omega = 0.01$. In this specific example the observed universe epoch sequence is reproduced.*
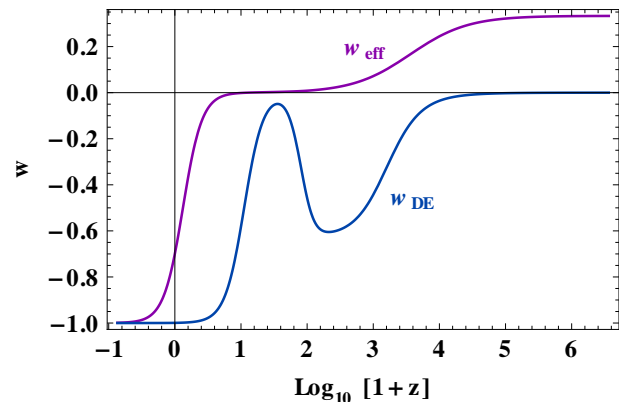

FIG. 3: *Cosmological evolution of the total (w_{eff}) and of the dark-energy (w_{DE}) equation-of-state parameters, as a function of the redshift z, for the parameter choice $\alpha_3 = 1$, $\alpha_4 = 0.115$ and $\omega = 0.01$. In this specific example, dark energy starts from a dust-like behavior, resulting in a cosmological-constant-like one at the current epoch, which drives the observed universe acceleration.*

dominates, both w_{eff} and w_{DE} coincide at the cosmological constant value.

B. Bouncing and turnaround solutions

In this subsection we examine the bounce realization in the scenario at hand. Bouncing cosmologies in general have gained significant interest, since they offer an alternative paradigm free of the "Big-Bang singularity", as well as of the horizon, flatness and monopole problems [96]. The basic condition for the bounce is the Hubble parameter to change sign from negative to positive, while

its time-derivative to be positive. Additionally, when the Hubble parameter changes from positive to negative, while its time-derivative is negative, we have the realization of a cosmological turnaround. Such a behavior implies the Null Energy Condition (NEC) violation around the bounce (or the turnaround), which is nontrivial in the context of General Relativity [97], leading in general to ghost degrees of freedom [98, 99]. However, a safe bounce evolution, free of ghosts and instabilities, is possible in various modified gravity constructions [100, 101]. Clearly, observing the Friedmann equation (19) of the present scenario of quasi-Dilaton non-linear massive gravity, we deduce that the basic bounce (or turnaround) realization condition can be easily fulfilled.

In order to quantify the above discussion, and assuming standard dust matter and standard radiation, we rewrite the Friedmann equation (19) in the following form:

$$\frac{H^2}{H_0^2} = \frac{\Omega_{m0}a^{-3} + \Omega_{r,0}a^{-4} - \frac{m^2}{H_0^2}G_1}{1 - \frac{\omega}{6}\left(1 - 4\frac{G_2}{\xi G_2'}\right)^2}, \qquad (40)$$

where Ω_{m0} and $\Omega_{r,0}$ are the values of the corresponding density parameters at present ($a_0 = 1$), and H_0 the current Hubble parameter. Furthermore, since the constraint equation (18) provides ξ as a function of the scale factor, we can finally write (40) as

$$\frac{H^2}{H_0^2} = V(a), \qquad (41)$$

with $C, \alpha_3, \alpha_4, \omega, m$ and $\Omega_{m0}, \Omega_{r,0}$ as parameters. In summary, examining the behavior of the above known $V(a)$, we can find possible bouncing points at $a = a_B$ (when $V(a_B) = 0$ and $H_0^{-1}\frac{d}{dt}\sqrt{V(a_B)} > 0$), or possible turnaround points at $a = a_T$ (when $V(a_T) = 0$ and $H_0^{-1}\frac{d}{dt}\sqrt{V(a_T)} < 0$). This procedure is straightforward, and thus we deduce that both bounce and turnaround are possible in the scenario of quasi-Dilaton non-linear massive gravity. Although one can apply it in full generality, obtaining exact results, since the involved expressions for a_B and a_T are lengthy in the following we explicitly apply it in a specific simple example, where simple expressions can be easily obtained.

First of all, since a bounce occurs in the early universe, that is at small scale factors ($a \ll a_0 = 1$), from (18) we deduce that at the bounce $\xi \gg 1$. On the other hand, since a turnaround occurs in the late universe, that is at $a \gg a_0 = 1$, from (18) we deduce that at the turnaround $G_2(\xi) = 0$.

Let us for simplicity investigate the simplest case $\alpha_3 = \alpha_4 = 0$. At the bounce region from (15) we obtain $G_1 \approx \xi^2$ and $G_2 \approx -\xi^2 = C/a^4$, which implies that $C < 0$. Therefore, from (40) we deduce that

$$\frac{H^2}{H_0^2} \approx \frac{\Omega_{m0}\left(1 - \frac{a_B}{a}\right)a^{-3}}{1 - \omega/6}, \qquad (42)$$

with

$$a_B = -\frac{\Omega_{r,0} + \frac{Cm^2}{H_0^2}}{\Omega_{m0}}, \qquad (43)$$

and therefore

$$\frac{\dot{H}}{H_0^2} \approx -\frac{3\Omega_{m0}a^{-3}\left(3 - 4\frac{a_B}{a}\right)}{6 - \omega}. \qquad (44)$$

From the above relations we easily acquire that $H(a_B) = 0$ and moreover for $\omega < 6$, which is always the case in interesting cosmology as we discussed in section III, we have $\dot{H}(a_B) > 0$. Thus, we immediately see that the bounce is obtained at $a = a_B$. Finally, note that obviously we require $a_B > 0$, which gives an additional constraint on the parameters, namely $Cm^2/H_0^2 < -\Omega_{r,0}$.

Let us now see whether a turnaround can occur in this simplest case $\alpha_3 = \alpha_4 = 0$. For $a \gg 1$, that is for $G_2 = 0$, we have $\xi = 0$ or $\xi = 1$. Obviously, since ξ varies, the above conditions are quite difficult to be obtained, and thus contrary to the bounce, the turnaround is harder to be acquired. In the $\xi = 0$ case from (15) we acquire $G_1 = 2$, and therefore from (40) we obtain

$$\frac{H^2}{H_0^2} \approx \frac{\Omega_{m0}\left(a^{-3} - a_T^{-3}\right)}{1 - \frac{3\omega}{2}} \qquad (45)$$

with

$$a_T = \left(\frac{H_0^2\Omega_{m0}}{2m^2}\right)^{1/3}, \qquad (46)$$

where for simplicity we neglected the radiation term since at large scale factors the matter term will dominate in general. Thus,

$$\frac{\dot{H}}{H_0^2} \approx -\frac{\frac{3}{2}\Omega_{m0}a^{-3}}{1 - \frac{3\omega}{2}}. \qquad (47)$$

From these relations we deduce that $H(a_T) = 0$, while $\dot{H}(a_T) < 0$ for $\omega < 2/3$, which is the turnaround condition (note that in this case $a_T > 0$ always). Concerning the second case $\xi = 1$, from (15) we acquire $G_1 = 0$, and thus $H^2 \propto \Omega_{m0}a^{-3} + \Omega_{r,0}a^{-4}$ which has no roots, and hence we deduce that in this case the turnaround is impossible.

Finally, note that for $\omega < 2/3$ and $Cm^2/H_0^2 < -\Omega_{r,0}$, in the above simplest example we obtain both a bounce and a turnaround, which is the realization of cyclic cosmology [96]. In Fig. 4, we present such a behavior, by numerically evolving the exact cosmological equations for parameter choices that satisfy the above conditions.

One can straightforwardly perform the above analysis for general α_3 and α_4, extracting the corresponding conditions for a bounce and turnaround, or for their simultaneous realization, that is for cyclic cosmology. In summary, as we can see, in the scenario of quasi-Dilaton non-linear massive gravity the above alternative cosmological evolutions can be easily obtained, in agreement with the general case of extended non-linear massive gravity [77].
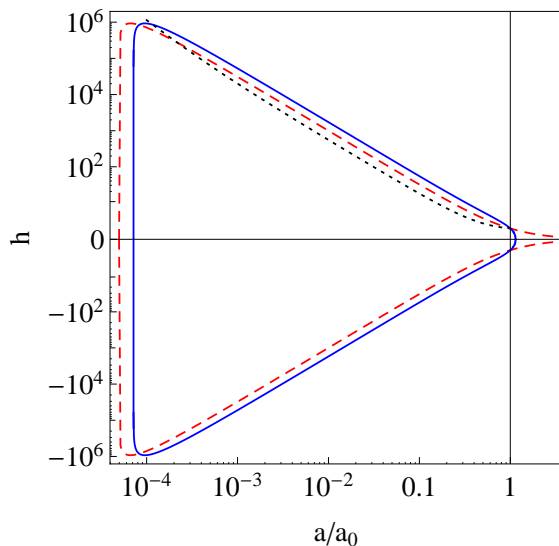
FIG. 4: *The rescaled Hubble function $h = H/H_0$ as a function of a/a_0, for $\alpha_3 = \alpha_4 = 0$, $\Omega_m = 0.3$, $\Omega_r = 10^{-4}$, $C = -1.1 \times 10^{-4}$ and $m = H_0$. In this case we obtain two evolution branches, one corresponding to a cyclic Universe (blue line) and a second branch corresponding to a Universe with a past bounce (red dashed curve). The dotted line corresponds to ΛCDM scenario with the same cosmological parameters.*

V. OBSERVATIONAL CONSTRAINTS

Having analyzed the cosmological behavior in the scenario of quasi-Dilaton non-linear massive gravity, we now proceed to investigate the observational constraints on the model parameters. We use observational data from Type Ia Supernovae (SNIa), Baryon Acoustic Oscillations (BAO), and Cosmic Microwave Background (CMB), and as usual we work in the standard units suitable for observational comparisons, namely setting $8\pi G = c = 1$.

In order to perform the analysis, and assuming standard dust matter and standard radiation ($w_r = 1/3$), we re-write the Friedmann equation (19) in the form of (40), and we use the redshift z instead of the scale factor. Thus, the scenario at hand has the parameters $C, \alpha_3, \alpha_4, \omega, m$ and Ω_{m0} or Ω_{DE0} (we fix H_0 by its 7-year WMAP best-fit value [102]). The details of the combined SNIa+CMB+BAO analysis are given in the Appendix, and here we present the constructed likelihood contours, discussing their structure.

Let us first examine the constraints on α_3 and α_4. In Fig. 5 we depict the 1σ and 2σ $\alpha_3 - \Omega_{m0}$ and $\alpha_4 - \Omega_{m0}$ contours respectively. As we observe, the parameters α_3 and α_4 are not constrained by the observational data, that is they only slightly affect the cosmological evolution. This behavior verifies the results obtained in section III, where the stable late-time solutions were found to be independent of α_3, α_4, as well as those obtained in section IV, where at intermediate times different values of α_3, α_4 lead to nearly the same cosmology. This is in agreement
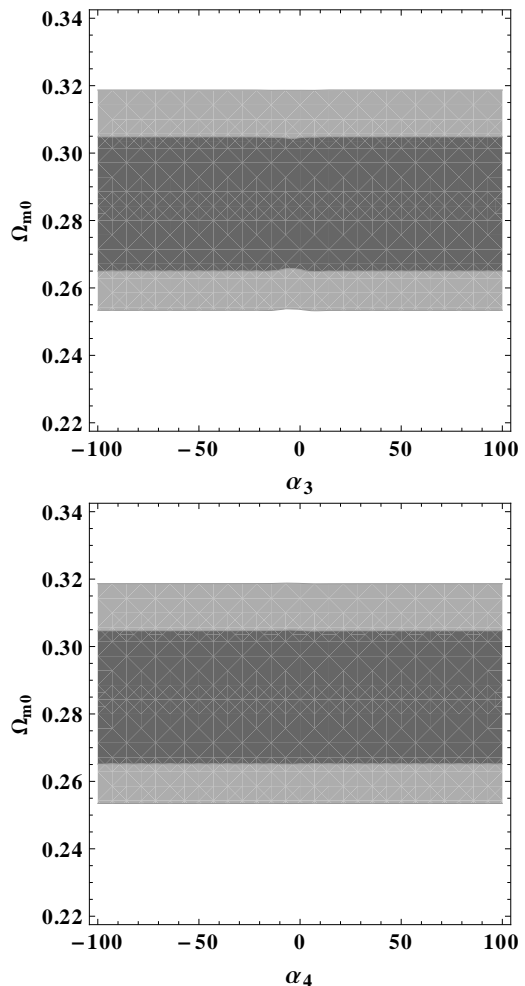

FIG. 5: *Top: The 1σ (dark) and 2σ (light) likelihood contours in the $\alpha_3 - \Omega_{m0}$ plane, using SNIa, BAO and CMB data, for $\alpha_4 = 1$. Bottom: The 1σ (dark) and 2σ (light) likelihood contours in the $\alpha_4 - \Omega_{m0}$ plane using SNIa, BAO and CMB data, for $\alpha_3 = 1$. Both graphs are for $\omega = 0.01$.*

with the analysis of [57] for the simple non-linear massive gravity. Finally, concerning Ω_{m0} we obtain a best fit value approximately at 0.284 for both α_3 and α_4.

In order to extract the constraints on the quasi-dilaton parameter ω, and knowing the above results on the insignificant role of α_3 and α_4, we fix them to $\alpha_3 = 1, \alpha_4 = 0.115$ (alternatively we could consider them as free parameters, marginalizing with flat priors, but the results would be almost the same). In Fig. 6 we depict the 1σ and 2σ likelihood contours in the $\omega - \Omega_{m0}$ plane. As we observe ω is significantly constrained by observations, with its best-fit value being $\omega = 0.0878$. Note also that negative ω values are also allowed, leading the dilaton to behave as a phantom field. This strong constraining behavior verifies the results of the dynamical analysis of section III, where in the case of interesting late-time cosmology ω was bounded. Finally, we mention that apart
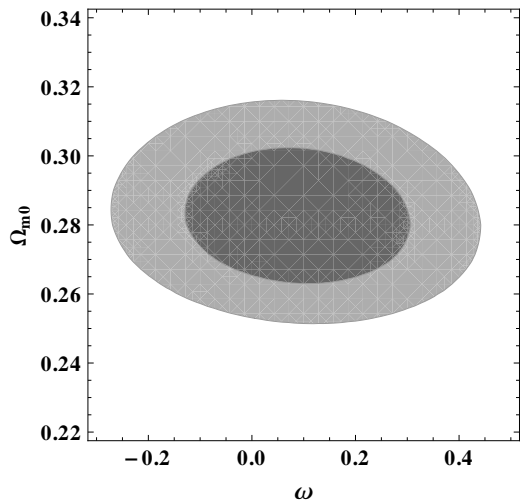
FIG. 6: *The 1σ (dark) and 2σ (light) likelihood contours in the $\omega - \Omega_{m0}$ plane using SNIa, BAO and CMB data, for $\alpha_3 = 1$ and $\alpha_4 = 0.115$.*

from the above late-time observational data, we could use Big Bang Nucleosynthesis arguments [103] in order to impose constraints on Ω_{DE}, that is to ω, however these would be weaker than the ones obtained above.

Finally, we examine the constraints on the graviton mass m, and in Fig. 7 we depict the corresponding likelihood contours in relation to Ω_{m0}. From this figure we observe that the ratio m/H_0 is highly constrained, and in particular we verify the expected result $m \sim H_0$, although with m being a bit smaller than H_0. In particular, the best-fit values are $m/H_0 = 0.44775$ and $\Omega_{m0} = 0.2794$.
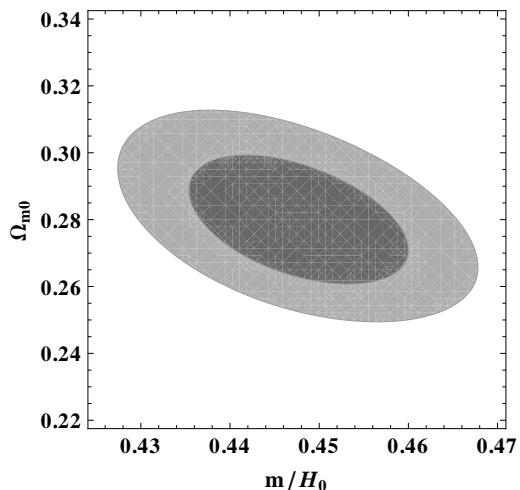

FIG. 7: *The 1σ (dark) and 2σ (light) likelihood contours in the $m/H_0 - \Omega_{m0}$ plane using SNIa, BAO and CMB data, for $\alpha_3 = 1$ and $\alpha_4 = 0.115$.*

VI. CONCLUSIONS

In this work we investigated the cosmological dynamics of the quasi-Dilaton non-linear massive gravity. In such a gravitational construction, one uses the usual non-linear massive gravity [6, 7], which is free of ghosts and couples it with a dilaton field [78]. Apart from the matter and radiation sectors in this scenario we obtain an effective dark energy, constituted from the massive graviton and the dilaton contributions. The corresponding cosmological behaviors proves to be very interesting.

First we performed a detailed dynamical analysis, and we extracted the stable late-time solutions, that is solutions that always attract the universe at late times. Furthermore, in each of these states we have calculated the corresponding observables, such as the dark-energy and matter density parameters, the dark-energy equation-of-state parameter, and the deceleration parameter. We showed that the scenario at hand gives rise to a dark-energy dominated accelerating universe where dark energy behaves like cosmological constant.

Leaving aside the good asymptotic behavior at late times, the cosmological behavior of quasi-Dilaton non-linear massive gravity has also interesting intermediate epochs *a la* thermal history. We have shown that the observed history of the universe can be easily obtained in a parameter region, that is the expected sequence of radiation, matter, and dark energy epochs. As a bonus, we found conditions on the parameters that might lead to bouncing, turnaround or cyclic behavior which is an interesting feature of the scenario under consideration.

Last but not least, we have imposed observational constraints on the model parameters. We carried out model-fitting using data from Type Ia Supernovae (SNIa), Baryon Acoustic Oscillations (BAO), and Cosmic Microwave Background (CMB) observations. Constructing the corresponding contour plots we deduced that the usual α_3 and α_4 parameters of non-linear massive gravity are not constrained by current data at the background level. However, the dilaton-coupling parameter is significantly constrained at a narrow window around zero. Finally, as expected, the graviton mass is found to be of the same order of the present value of the Hubble parameter.

From the aforesaid, we find that cosmology of the quasi-Dilaton non-linear massive gravity is interesting and is in agreement with the observed universe behavior. In our opinion, it can be a good candidate for the description of universe. However, in order to obtain additional indications of the validity and consistency with observations, of the scenario at hand, apart from the background analysis that was the base of this work, we should proceed to a detailed examination of the perturbations. In particular, one would need to investigate the perturbation-related observables such as the spectral index, the tensor-to-scalar ratio, and the non-gaussianity estimators. Such an analysis lies beyond the goal of the present work and it is left for future investigations.

Appendix: Observational data and constraints

Here we briefly review the sources of observational constraints used in this manuscript, namely Type Ia Supernovae constraints, Baryon Acoustic Oscillations (BAO) and Cosmic Microwave Background (CMB). The total χ^2 defined as

$$\chi^2 = \chi^2_{SN} + \chi^2_{BAO} + \chi^2_{CMB}, \tag{A.1}$$

where the individual χ^2_i for every data set is calculated as follows.

a. Type Ia Supernovae constraints

In order to incorporate Type Ia constraints we use the Union2.1 data compilation [104] of 580 data points. The relevant observable is the distance modulus μ which is defined as $\mu = m - M = 5 \log D_L + \mu_0$, where m and M are the apparent and absolute magnitudes of the Supernovae, $D_L(z)$ is the luminosity distance $D_L(z) = (1+z) \int_0^z \frac{H_0 dz'}{H(z')}$ and $\mu_0 = 5 \log \left(\frac{H_0^{-1}}{M_{pc}} \right) + 25$ is a nuisance parameter that should be marginalized. The corresponding χ^2 writes as

$$\chi^2_{SN}(\mu_0, \theta) = \sum_{i=1}^{580} \frac{[\mu_{th}(z_i, \mu_0, \theta) - \mu_{obs}(z_i)]^2}{\sigma_\mu(z_i)^2}, \tag{A.2}$$

where μ_{obs} denotes the observed distance modulus while μ_{th} the theoretical one, and σ_μ is the uncertainty in the distance modulus. Additionally, θ denotes any parameter of the specific model at hand. Finally, marginalizing μ_0 following [105] we obtain

$$\chi^2_{SN}(\theta) = A(\theta) - \frac{B(\theta)^2}{C(\theta)}, \tag{A.3}$$

with

$$A(\theta) = \sum_{i=1}^{580} \frac{[\mu_{th}(z_i, \mu_0 = 0, \theta) - \mu_{obs}(z_i)]^2}{\sigma_\mu(z_i)^2} \tag{A.4}$$

$$B(\theta) = \sum_{i=1}^{580} \frac{\mu_{th}(z_i, \mu_0 = 0, \theta) - \mu_{obs}(z_i)}{\sigma_\mu(z_i)^2} \tag{A.5}$$

$$C(\theta) = \sum_{i=1}^{580} \frac{1}{\sigma_\mu(z_i)^2}. \tag{A.6}$$

b. Baryon Acoustic Oscillation constraints

We use BAO data from [106–109], that is of $\frac{d_A(z_\star)}{D_V(Z_{BAO})}$, where $z_\star$ is the decoupling time given by $z_\star \approx 1091$, the co-moving angular-diameter distance $d_A(z) = \int_0^z \frac{dz'}{H(z')}$ and $D_V(z) = \left(d_A(z)^2 \frac{z}{H(z)} \right)^{\frac{1}{3}}$ is the dilation scale [110]. th required data are depicted in Table II.

In order to calculate χ^2_{BAO} for BAO data we follow the procedure described in [111], where it is defined as,

$$\chi^2_{BAO} = X^T_{BAO} C^{-1}_{BAO} X_{BAO}, \tag{A.7}$$

with

$$X_{BAO} = \begin{pmatrix} \frac{d_A(z_\star)}{D_V(0.106)} - 30.95 \\ \frac{d_A(z_\star)}{D_V(0.2)} - 17.55 \\ \frac{d_A(z_\star)}{D_V(0.35)} - 10.11 \\ \frac{d_A(z_\star)}{D_V(0.44)} - 8.44 \\ \frac{d_A(z_\star)}{D_V(0.6)} - 6.69 \\ \frac{d_A(z_\star)}{D_V(0.73)} - 5.45 \end{pmatrix} \tag{A.8}$$

and the inverse covariance matrix reads as

$$C^{-1} = \begin{pmatrix} 0.48435 & -0.101383 & -0.164945 & -0.0305703 & -0.097874 & -0.106738 \\ -0.101383 & 3.2882 & -2.45497 & -0.0787898 & -0.252254 & -0.2751 \\ -0.164945 & -2.45499 & 9.55916 & -0.128187 & -0.410404 & -0.447574 \\ -0.0305703 & -0.0787898 & -0.128187 & 2.78728 & -2.75632 & 1.16437 \\ -0.097874 & -0.252254 & -0.410404 & -2.75632 & 14.9245 & -7.32441 \\ -0.106738 & -0.2751 & -0.447574 & 1.16437 & -7.32441 & 14.5022 \end{pmatrix}. \tag{A.9}$$

c. CMB constraints

z_{BAO}	0.106	0.2	0.35	0.44	0.6	0.73
$\frac{d_A(z_\star)}{D_V(Z_{BAO})}$	30.95 ± 1.46	17.55 ± 0.60	10.11 ± 0.37	8.44 ± 0.67	6.69 ± 0.33	5.45 ± 0.31

TABLE II: Values of $\frac{d_A(z_\star)}{D_V(Z_{BAO})}$ for different values of z_{BAO}.

We use the CMB shift parameter

$$R = H_0 \sqrt{\Omega_{m0}} \int_0^{1089} \frac{dz'}{H(z')} \qquad (A.10)$$

following [102]. The corresponding χ^2_{CMB} is defined as,

$$\chi^2_{CMB}(\theta) = \frac{(R(\theta) - R_0)^2}{\sigma^2}, \qquad (A.11)$$

with $R_0 = 1.725 \pm 0.018$ and $R(\theta)$ [102].

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